

Strong rainbow connection numbers of toroidal meshes

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Abstract

In 2011, Li et al. [5] obtained an upper bound of the strong rainbow connection number of an r -dimensional undirected toroidal mesh. In this paper, this bound is improved. As a result, we give a negative answer to their problem.

Key words: toroidal mesh; (strong) rainbow path; (strong) rainbow connection number; Cayley graph.

1 Introduction

All graphs considered in this paper are finite, connected and simple. We refer to the book [2] for graph theory notation and terminology not described here. Let Γ be a graph. Denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set of Γ , respectively. A sequence of distinct vertices is a path if any two consecutive vertices are adjacent. A path $P : (v_1, v_2, \dots, v_k)$ is a cycle if v_1 is adjacent to v_k , denoted by C_k . The distance, $d(u, v)$, between vertices u and v is equal to the length of a shortest path connecting u and v . The diameter of Γ , $d(\Gamma)$, is the maximum distance between two vertices in Γ over all pairs of vertices.

Define a k -edge-coloring $\zeta : E(\Gamma) \rightarrow \{1, 2, \dots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path is *rainbow* if no two edges of it are colored the same. A path from u to v is called a *strong rainbow path* if it's a rainbow path with length $d(u, v)$. If any two distinct vertices u and v of Γ are connected by a (strong) rainbow path, then Γ is called *(strong) rainbow-connected* under the coloring ζ , and ζ is called a *(strong) rainbow k -coloring* of Γ . The *(strong) rainbow connection number* of Γ , denoted by $(\text{src}(\Gamma)) \text{rc}(\Gamma)$, is the minimum k for which there exists a (strong) rainbow k -coloring of Γ . Clearly, we have $d(\Gamma) \leq \text{rc}(\Gamma) \leq \text{src}(\Gamma)$.

The (strong) rainbow connection number of a graph was first introduced by Chartrand et al. [3]. Ananth and Nasre [1] proved that, for every integer $k \geq 3$, deciding whether $\text{src}(\Gamma) \leq k$ is NP-hard even when Γ is bipartite. (Strong) rainbow connection numbers of some special graphs have been studied in the literature, such as outerplanar graphs [4], Cayley graphs [5, 8], line graphs [6], power graphs [7], undirected double-loop networks [9] and non-commuting graphs [10].

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The Cartesian product of two graphs Γ and Λ is the graph $\Gamma \square \Lambda$ whose vertex set is the set $\{\gamma\lambda \mid \gamma \in V(\Gamma), \lambda \in V(\Lambda)\}$, and two vertices $\gamma\lambda, \gamma'\lambda'$ are adjacent if $\lambda = \lambda'$ and $\{\gamma, \gamma'\} \in E(\Gamma)$ or if $\gamma = \gamma'$ and $\{\lambda, \lambda'\} \in E(\Lambda)$. The Cartesian product operation is commutative and associative, hence the Cartesian product of more factors is well-defined. The graph $C_{n_1} \square \cdots \square C_{n_r}$ is an r -dimensional undirected toroidal mesh, where $n_k \geq 2$ for $1 \leq k \leq r$.

In 2011, Li et al. proved the following theorem and proposed an open problem.

Theorem 1.1 [5, Corollary 2] *Let C_{n_k} , $n_k \geq 2$, $1 \leq k \leq r$ be cycles. Then*

$$\sum_{1 \leq k \leq r} \left\lfloor \frac{n_k}{2} \right\rfloor \leq \text{rc}(C_{n_1} \square \cdots \square C_{n_r}) \leq \text{src}(C_{n_1} \square \cdots \square C_{n_r}) \leq \sum_{1 \leq k \leq r} \left\lceil \frac{n_k}{2} \right\rceil.$$

Moreover, if n_k is even for every $1 \leq k \leq r$, then

$$\text{rc}(C_{n_1} \square \cdots \square C_{n_r}) = \text{src}(C_{n_1} \square \cdots \square C_{n_r}) = \sum_{1 \leq k \leq r} \frac{n_k}{2}.$$

Problem 1.1 [5, Remark 2] *Given an Abelian group G and an inverse closed minimal generating set $S \subseteq G \setminus 1$ of G , is it true that*

$$\text{src}(C(G, S)) = \text{rc}(C(G, S)) = \sum_{a \in S^*} \left\lceil \frac{|a|}{2} \right\rceil?$$

where $S^* \subseteq S$ is a minimal generating set of G .

In this paper, we improve the upper bound of $\text{src}(C_{n_1} \square \cdots \square C_{n_r})$ in Theorem 1.1. Our main result is listed below.

Theorem 1.2 *Let C_{n_k} , $n_k \geq 2$, $1 \leq k \leq r$ be cycles. Then*

$$\text{src}(C_{n_1} \square \cdots \square C_{n_r}) \leq \begin{cases} \left\lceil \frac{n_1 + \cdots + n_r - \mu}{2} \right\rceil, & 0 \leq \mu \leq \left\lfloor \frac{r}{2} \right\rfloor; \\ \left\lceil \frac{n_1 + \cdots + n_r - r + \mu}{2} \right\rceil, & \left\lfloor \frac{r}{2} \right\rfloor + 1 \leq \mu \leq r, \end{cases} \quad (1)$$

where μ is the number of even numbers among n_1, \dots, n_r .

Note that an r -dimensional undirected toroidal mesh is a Cayley graph. As a result, Theorem 1.2 gives a negative answer to Problem 1.1.

2 Preliminary results

In this section, we will introduce some useful results for the strong rainbow connection numbers of graphs.

Lemma 2.1 [3, Proposition 2.1] *For each integer $n \geq 4$, $\text{rc}(C_n) = \text{src}(C_n) = \left\lceil \frac{n}{2} \right\rceil$.*

We make the following simple observation, which we will use repeatedly.

Observation 2.2 *Let Γ and Λ be two connected graphs. Then*

$$\text{src}(\Gamma) \leq \text{src}(\Gamma \square \Lambda) \leq \text{src}(\Gamma) + \text{src}(\Lambda).$$

Lemma 2.3 *For each integer $n \geq 3$, $\text{src}(C_n \square C_2) = \lceil \frac{n+1}{2} \rceil$.*

Proof. Write $(0, 1, \dots, n-1)$ for C_n and $(0, 1)$ for C_2 . Since the diameter of $C_n \square C_2$ is $\lceil \frac{n+1}{2} \rceil$, it suffices to show that $\text{src}(C_n \square C_2) \leq \lceil \frac{n+1}{2} \rceil$. We only need to construct a strong rainbow $\lceil \frac{n+1}{2} \rceil$ -coloring. Now we divide our discussion into two cases.

Case 1 $n = 2k$.

Define an edge-coloring f_1 of the graph $C_{2k} \square C_2$ by

$$f_1(e) = \begin{cases} k, & \text{if } e = \{i0, i1\}; \\ i, & \text{if } e = \{ij, (i+1)j\}, 0 \leq i \leq k-1; \\ i-k, & \text{if } e = \{ij, (i+1)j\}, k \leq i \leq 2k-2; \\ k-1, & \text{if } e = \{(2k-1)j, 0j\}. \end{cases}$$

For illustration, we give a strong rainbow 5-coloring of $C_8 \square C_2$ in Figure 1.

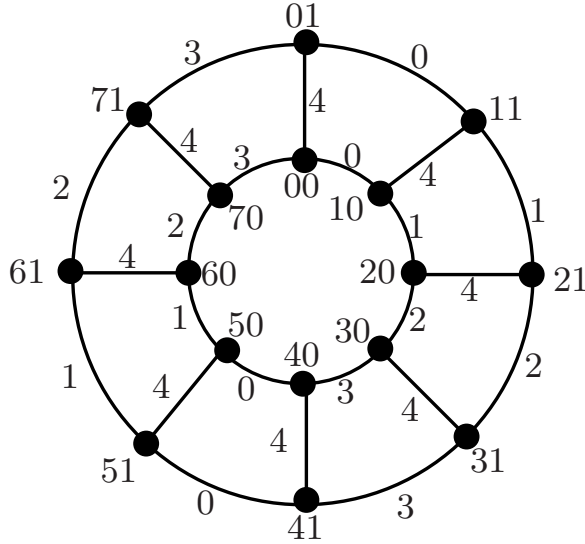


Figure 1: A strong rainbow 5-coloring of $C_8 \square C_2$

Note that any path from u to v in Table 1 is a strong rainbow path under the coloring f_1 . It follows that f_1 is a strong rainbow $(k+1)$ -coloring.

Case 2 $n = 2k+1$.

Define an edge-coloring f_2 of the graph $C_{2k+1} \square C_2$ by

$$f_2(e) = \begin{cases} i, & \text{if } e = \{i0, i1\}, 1 \leq i \leq k; \\ i-k-1, & \text{if } e = \{i0, i1\}, k+1 \leq i \leq 2k; \\ k, & \text{if } e = \{00, 01\}; \\ i, & \text{if } e = \{ij, (i+1)j\}, 0 \leq i \leq k; \\ i-k-1, & \text{if } e = \{ij, (i+1)j\}, k+1 \leq i \leq 2k-1; \\ k-1, & \text{if } e = \{(2k)j, 0j\}. \end{cases}$$

Table 1: A path from u to v in $C_{2k} \square C_2$

u	v	Condition	A path from u to v
is	js	$1 \leq j - i \leq k$	$(is, (i+1)s, \dots, js)$
is	js	$j - i \geq k+1$	$(js, (j+1)s, \dots, (2k-1)s, 0s, 1s, \dots, is)$
is	js	$1 \leq i - j \leq k$	$(js, (j+1)s, \dots, is)$
is	js	$i - j \geq k+1$	$(is, (i+1)s, \dots, (2k-1)s, 0s, 1s, \dots, js)$
$i0$	$j1$	$1 \leq j - i \leq k$	$(i0, i1, (i+1)1, \dots, j1)$
$i0$	$j1$	$j - i \geq k+1$	$(j0, j1, (j+1)1, \dots, (2k-1)1, 01, 11, \dots, i1)$
$i0$	$j1$	$1 \leq i - j \leq k$	$(j0, j1, (j+1)1, \dots, i1)$
$i0$	$j1$	$i - j \geq k+1$	$(i0, i1, (i+1)1, \dots, (2k-1)1, 01, 11, \dots, j1)$

For illustration, we give a strong rainbow 4-coloring of $C_7 \square C_2$ in Figure 2.

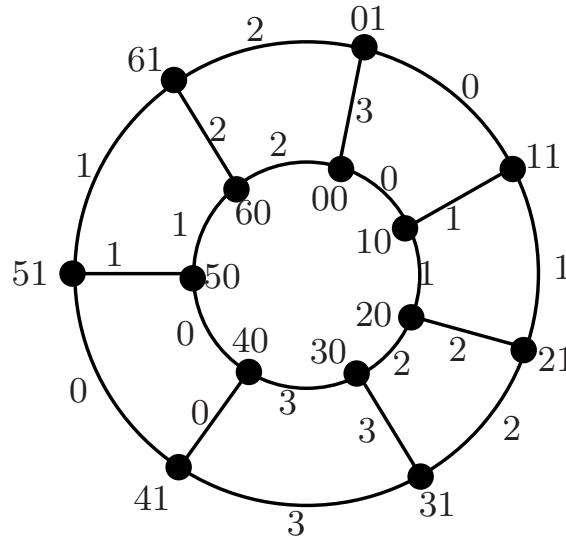


Figure 2: A strong rainbow 4-coloring of $C_7 \square C_2$

Note that any path from u to v in Table 2 is a strong rainbow path under the coloring f_2 . It follows that f_2 is a strong rainbow $(k+1)$ -coloring. \square

In the graph $\Gamma \square \Lambda$, we write Γy for $\Gamma \square \{y\}$, where $y \in V(\Lambda)$. The union of graphs Γ and Λ is the graph $\Gamma \cup \Lambda$ with vertex set $V(\Gamma) \cup V(\Lambda)$ and edge set $E(\Gamma) \cup E(\Lambda)$.

Proposition 2.4 *Let Γ be a connected graph. Then*

$$\text{src}(\Gamma \square C_n) \leq \left\lceil \frac{n-2}{2} \right\rceil + \text{src}(\Gamma \square C_2), n \geq 3. \quad (2)$$

Proof. Write (x_1, x_2, \dots, x_n) for C_n . Meanwhile, we use $E[\Gamma x_i, \Gamma x_j]$ to denote the edge set between Γx_i and Γx_j .

By Observation 2.2, we have

$$\text{src}(\Gamma \square C_3) \leq \text{src}(C_3) + \text{src}(\Gamma) = 1 + \text{src}(\Gamma) \leq 1 + \text{src}(\Gamma \square C_2).$$

Thus, (2) holds for $n = 3$. Now, we suppose that $n \geq 4$.

Let L_1 and L_2 be induced subgraphs of $\Gamma \square C_n$ whose vertex sets are $V(\Gamma x_1) \cup V(\Gamma x_n)$ and $V(\Gamma x_{\lfloor \frac{n}{2} \rfloor}) \cup V(\Gamma x_{\lfloor \frac{n}{2} \rfloor + 1})$ respectively. Then each L_i is isomorphic to

Table 2: A path from u to v in $C_{2k+1} \square C_2$

u	v	Condition	A path from u to v
is	js	$1 \leq j - i \leq k$	$(is, (i+1)s, \dots, js)$
is	js	$j - i \geq k + 1$	$(js, (j+1)s, \dots, (2k)s, 0s, 1s, \dots, is)$
is	js	$1 \leq i - j \leq k$	$(js, (j+1)s, \dots, is)$
is	js	$i - j \geq k + 1$	$(is, (i+1)s, \dots, (2k)s, 0s, 1s, \dots, js)$
00	$j1$	$0 \leq j \leq k$	$(00, 01, 11, \dots, j1)$
00	$j1$	$k+1 \leq j \leq 2k$	$(00, 01, (2k)1, \dots, j1)$
10	$j1$	$1 \leq j \leq k+1$	$(10, 20, \dots, j0, j1)$
10	$j1$	$j = 0$ or $k+2 \leq j \leq 2k$	$(10, 00, 01, (2k)1, \dots, j1)$
$i0$	$j1$	$2 \leq i \leq k-1$ and $0 \leq j \leq i$	$(i0, i1, (i-1)1, \dots, j1)$
$i0$	$j1$	$2 \leq i \leq k-1$ and $i+1 \leq j \leq i+k$	$(i0, (i+1)0, \dots, j0, j1)$
$i0$	$j1$	$2 \leq i \leq k-1$ and $i+k+1 \leq j \leq 2k$	$(i0, (i-1)0, \dots, 00, 01, (2k)1, (2k-1)1, \dots, j1)$
$k0$	$j1$	$0 \leq j \leq k$	$(k0, k1, (k-1)1, \dots, j1)$
$k0$	$j1$	$k+1 \leq j \leq 2k$	$(k0, (k+1)0, \dots, j0, j1)$
$i0$	$j1$	$k+1 \leq i \leq 2k-1$ and $0 \leq j \leq i-k-1$	$(i0, (i+1)0, \dots, 00, 01, \dots, j1)$
$i0$	$j1$	$k+1 \leq i \leq 2k-1$ and $i-k \leq j \leq i$	$(i0, i1, (i-1)1, \dots, j1)$
$i0$	$j1$	$k+1 \leq i \leq 2k-1$ and $i+1 \leq j \leq 2k$	$(i0, (i+1)0, \dots, j0, j1)$
$(2k)0$	$j1$	$0 \leq j \leq k-1$	$((2k)0, 00, 01, \dots, j1)$
$(2k)0$	$j1$	$k \leq j \leq 2k$	$((2k)0, (2k)1, (2k-1)1, \dots, j1)$

$\Gamma \square C_2$. Let $S_0 = \{1, \dots, \lceil \frac{n-2}{2} \rceil\}$. Suppose that $f_1 : E(\Gamma) \rightarrow S_1$ is a strong rainbow $\text{src}(\Gamma)$ -coloring of Γ , and $f_{2,i} : E(L_i) \rightarrow S_2$ is a strong rainbow $\text{src}(\Gamma \square C_2)$ -coloring of L_i for $1 \leq i \leq 2$, where $S_0 \cap S_2 = \emptyset$. By Observation 2.2, we may assume that $S_1 \subseteq S_2$. Define an edge-coloring $f_3 : E(\Gamma \square C_n) \rightarrow S_0 \cup S_2$ by

$$f_3(e) = \begin{cases} i, & \text{if } e \in E[\Gamma x_i, \Gamma x_{i+1}], 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1; \\ i - \lfloor \frac{n}{2} \rfloor, & \text{if } e \in E[\Gamma x_i, \Gamma x_{i+1}], \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n-1; \\ f_{2,i}(e), & \text{if } e \in E(L_i) \text{ for } 1 \leq i \leq 2; \\ f_1(\{y_1, y_2\}), & \text{if } e = \{y_1 x_i, y_2 x_i\} \in E[\Gamma x_i, \Gamma x_i], n \geq 5 \text{ and } i \in I, \end{cases}$$

where $I = \{2, 3, \dots, \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor + 2, \dots, \lfloor \frac{n}{2} \rfloor + 3, \dots, n-1\}$. For illustration of f_3 , see Figure 3.

Pick any two distinct vertices u and v of $\Gamma \square C_n$. Write $u = y_1 x_i$ and $v = y_2 x_j$. Without loss of generality, we may assume that $i \leq j$. We only need to show that there exists a strong rainbow path from u to v under f_3 . If $i = j$, the desired result is obvious. Assume that $i \neq j$. We divide our discussion into three cases.

Case 1 $1 \leq j - i \leq \lfloor \frac{n}{2} \rfloor$, and $2 \leq j \leq \lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n-1$.

Pick a strong rainbow path P_1 from $y_1 x_j$ to v in Γx_j . Then

$$(u = y_1 x_i, y_1 x_{i+1}, \dots, y_1 x_j) \cup P_1$$

is a desired strong rainbow path.

Case 2 $1 \leq j - i \leq \lfloor \frac{n}{2} \rfloor$, $j \geq \lfloor \frac{n}{2} \rfloor + 1$ and $i \leq \lfloor \frac{n}{2} \rfloor$.

Pick a strong rainbow path P_2 from $y_1 x_{\lfloor \frac{n}{2} \rfloor}$ to $y_2 x_{\lfloor \frac{n}{2} \rfloor + 1}$ in L_2 . Then

$$(u = y_1 x_i, y_1 x_{i+1}, \dots, y_1 x_{\lfloor \frac{n}{2} \rfloor}) \cup P_2 \cup (y_2 x_{\lfloor \frac{n}{2} \rfloor + 1}, y_2 x_{\lfloor \frac{n}{2} \rfloor + 2}, \dots, y_2 x_j = v)$$

is a desired strong rainbow path.

Case 3 $j - i \geq \lfloor \frac{n}{2} \rfloor + 1$.

Pick a strong rainbow path P_3 from $y_1 x_1$ to $y_2 x_n$ in L_1 . Then

$$(u = y_1 x_i, y_1 x_{i-1}, \dots, y_1 x_1) \cup P_3 \cup (y_2 x_n, y_2 x_{n-1}, \dots, y_2 x_{j+1}, y_2 x_j = v)$$

is a desired strong rainbow path.

As mentioned above, we obtain the desired result. \square

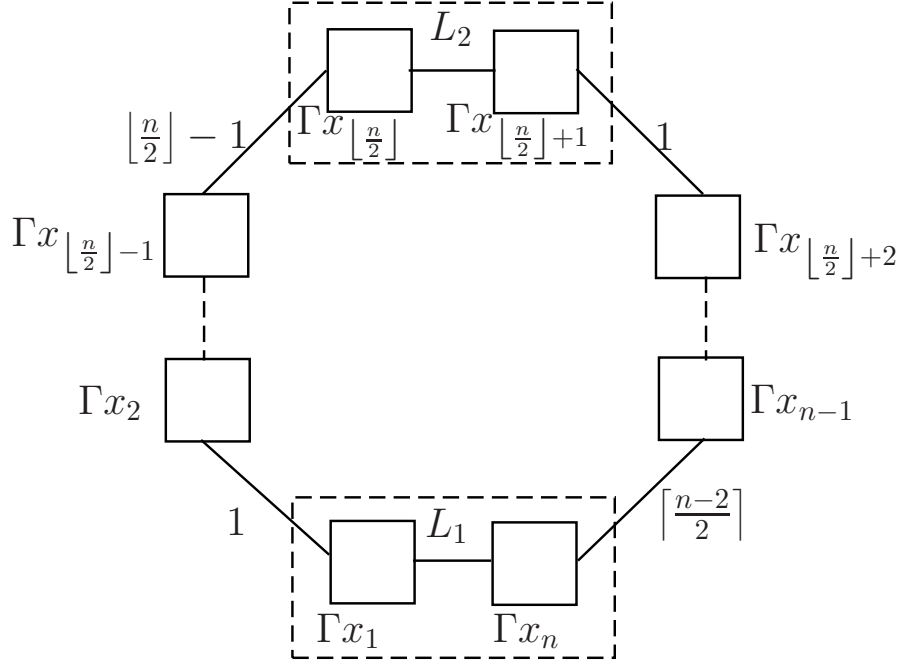


Figure 3: Illustration of f_3

3 Proof of Theorem 1.2

Proposition 3.1 *Let C_{n_k} , $n_k \geq 2$, $1 \leq k \leq r$ be cycles. Then*

$$\text{src}(C_{n_1} \square \cdots \square C_{n_r}) \leq \left\lceil \frac{n_1 + \cdots + n_r}{2} \right\rceil. \quad (3)$$

Proof. Without loss of generality, we may assume that $n_1 \geq n_2 \geq \cdots \geq n_r$. We distinguish two cases.

Case 1 $n_r \geq 3$.

We prove this proposition by induction on r . If $r = 1$, (3) is derived from Lemma 2.1. Suppose $r = 2$. If n_2 is even, then by Lemma 2.1 and Observation 2.2, (3) holds. If n_2 is odd, then by Proposition 2.4 and Lemma 2.3, we have

$$\text{src}(C_{n_1} \square C_{n_2}) \leq \left\lceil \frac{n_2 - 2}{2} \right\rceil + \text{src}(C_{n_1} \square C_2) = \left\lceil \frac{n_1 + n_2}{2} \right\rceil.$$

Now, Suppose $r \geq 3$.

If each n_i is odd, then

$$\begin{aligned}
& \text{src}(C_{n_1} \square \cdots \square C_{n_{r-1}} \square C_{n_r}) \\
& \leq \left\lceil \frac{n_r - 2}{2} \right\rceil + \text{src}((C_{n_1} \square \cdots \square C_{n_{r-2}}) \square (C_{n_{r-1}} \square C_2)) \quad (\text{by Proposition 2.4}) \\
& \leq \left\lceil \frac{n_r - 2}{2} \right\rceil + \text{src}(C_{n_1} \square \cdots \square C_{n_{r-2}}) + \text{src}(C_{n_{r-1}} \square C_2) \quad (\text{by Observation 2.2}) \\
& \leq \left\lceil \frac{n_r - 2}{2} \right\rceil + \left\lceil \frac{n_1 + \cdots + n_{r-2}}{2} \right\rceil + \left\lceil \frac{n_{r-1} + 1}{2} \right\rceil \quad (\text{by induction hypothesis and Lemma 2.3}) \\
& = \left\lceil \frac{n_1 + \cdots + n_r}{2} \right\rceil.
\end{aligned}$$

If n_i is even for some i , then

$$\begin{aligned}
& \text{src}(C_{n_1} \square \cdots \square C_{n_r}) \\
& \leq \text{src}(C_{n_1} \square \cdots \square C_{n_{i-1}} \square C_{n_{i+1}} \square \cdots \square C_{n_r}) + \text{src}(C_{n_i}) \quad (\text{by Observation 2.2}) \\
& \leq \left\lceil \frac{n_1 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_r}{2} \right\rceil + \left\lceil \frac{n_i}{2} \right\rceil \quad (\text{by induction hypothesis}) \\
& = \left\lceil \frac{n_1 + \cdots + n_r}{2} \right\rceil.
\end{aligned}$$

Case 2 $n_r = 2$.

Suppose s is the minimum positive integer such that $n_s = 2$.

Case 2.1 $s = 1$. By Observation 2.2, (3) is obtained.

Case 2.2 $s \geq 2$. In this case, we have

$$\begin{aligned}
& \text{src}(C_{n_1} \square \cdots \square C_{n_r}) \\
& \leq \text{src}(C_{n_1} \square \cdots \square C_{n_{s-1}}) + \text{src}(C_2 \square \cdots \square C_2) \quad (\text{by Observation 2.2}) \\
& \leq \left\lceil \frac{n_1 + \cdots + n_{s-1}}{2} \right\rceil + r - s + 1 \quad (\text{by Case 1}) \\
& = \left\lceil \frac{n_1 + \cdots + n_r}{2} \right\rceil.
\end{aligned}$$

Combining Case 1 and Case 2, we obtain the desired result. \square

Proof of Theorem 1.2: If $r = 1$, (1) is obvious. Now, suppose $r \geq 2$. We divide our discussion into three cases.

Case 1 $r = 2$.

If $n_1 = n_2 = 2$, Observation 2.2 implies that (1) holds. Now suppose $n_j \geq 3$, for some j . Without loss of generality, we may assume that $n_1 \geq 3$. By Proposition 2.4 and Lemma 2.3, we have

$$\text{src}(C_{n_1} \square C_{n_2}) \leq \left\lceil \frac{n_2 - 2}{2} \right\rceil + \text{src}(C_{n_1} \square C_2) = \left\lceil \frac{n_2 - 2}{2} \right\rceil + \left\lceil \frac{n_1 + 1}{2} \right\rceil. \quad (4)$$

If $\mu = 1$ and n_1 is even, then

$$\text{src}(C_{n_1} \square C_{n_2}) = \text{src}(C_{n_2} \square C_{n_1}) \leq \left\lceil \frac{n_1 - 2}{2} \right\rceil + \text{src}(C_{n_2} \square C_2) = \left\lceil \frac{n_1 + n_2 - 1}{2} \right\rceil.$$

Otherwise, (4) implies (1).

Case 2 $r = 3$.

If $\mu = 0$ or $\mu = 3$, (1) holds by Proposition 3.1. Now suppose $\mu = 1$ or $\mu = 2$. Without loss of generality, we assume that n_1 is even and n_3 is odd. By Observation 2.2, Lemma 2.1 and Case 1, we have

$$\text{src}(C_{n_1} \square C_{n_2} \square C_{n_3}) \leq \text{src}(C_{n_1}) + \text{src}(C_{n_2} \square C_{n_3}) \leq \left\lceil \frac{n_1 + n_2 + n_3 - 1}{2} \right\rceil.$$

Case 3 $r \geq 4$.

If $\mu = 0$ or $\mu = r$, Proposition 3.1 implies (1). In the following, we assume that n_1, \dots, n_μ are even and $n_{\mu+1}, \dots, n_r$ are odd.

Case 3.1 $1 \leq \mu \leq \lfloor \frac{r}{2} \rfloor - 1$.

$$\begin{aligned} & \text{src}(C_{n_1} \square \dots \square C_{n_r}) \\ & \leq \text{src}(C_{n_{2\mu+1}} \square \dots \square C_{n_r}) + \sum_{1 \leq s \leq \mu, s+t=2\mu+1} \text{src}(C_{n_s} \square C_{n_t}) \quad (\text{by Observation 2.2}) \\ & \leq \left\lceil \frac{n_{2\mu+1} + \dots + n_r}{2} \right\rceil + \sum_{1 \leq s \leq \mu, s+t=2\mu+1} \frac{n_s + n_t - 1}{2} \quad (\text{by Proposition 3.1 and Case 1}) \\ & = \left\lceil \frac{n_1 + \dots + n_r - \mu}{2} \right\rceil. \end{aligned}$$

Case 3.2 $\mu = \lfloor \frac{r}{2} \rfloor$.

$$\begin{aligned} & \text{src}(C_{n_1} \square \dots \square C_{n_r}) \\ & \leq \frac{1}{2}(1 + (-1)^{r+1})\text{src}(C_{n_r}) + \sum_{1 \leq s \leq \mu, s+t=2\mu+1} \text{src}(C_{n_s} \square C_{n_t}) \quad (\text{by Observation 2.2}) \\ & \leq \frac{1}{2}(1 + (-1)^{r+1}) \left\lceil \frac{n_r}{2} \right\rceil + \sum_{1 \leq s \leq \mu, s+t=2\mu+1} \frac{n_s + n_t - 1}{2} \quad (\text{by Proposition 3.1 and Case 1}) \\ & = \left\lceil \frac{n_1 + \dots + n_r - \mu}{2} \right\rceil. \end{aligned}$$

Case 3.3 $\lfloor \frac{r}{2} \rfloor + 1 \leq \mu \leq r - 1$.

$$\begin{aligned} & \text{src}(C_{n_1} \square \dots \square C_{n_r}) \\ & \leq \text{src}(C_{n_1} \square \dots \square C_{2\mu-r}) + \sum_{\mu+1 \leq t \leq r, s+t=2\mu+1} \text{src}(C_{n_s} \square C_{n_t}) \quad (\text{by Observation 2.2}) \\ & \leq \left\lceil \frac{n_1 + \dots + n_{2\mu-r}}{2} \right\rceil + \sum_{\mu+1 \leq t \leq r, s+t=2\mu+1} \frac{n_s + n_t - 1}{2} \quad (\text{by Proposition 3.1 and Case 1}) \\ & = \left\lceil \frac{n_1 + \dots + n_r - r + \mu}{2} \right\rceil. \end{aligned}$$

As mentioned above, we obtain the desired result. \square

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